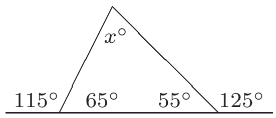
AMC WARM-UP PAPER
INTERMEDIATE PAPER 7
SOLUTIONS

1. From adjacent angles on a straight line and the angle sum of a triangle we have
$x + 65 + 55 = 180$
thus $x = 60$,

hence (C).



2.

$$\begin{aligned} 5n + 7 &> 100 \\ 5n &> 93 \\ n &> 18\frac{3}{5} \\ \text{thus} \qquad n &= 19, \end{aligned}$$

hence (B).

3. The lightest sweets will weigh at 30 per kilogram.
Thus the minimum weight for 240 is

$$\frac{240}{30} = 8 \text{ kg},$$

hence (C).

4. Barbara's share (in dollars) is

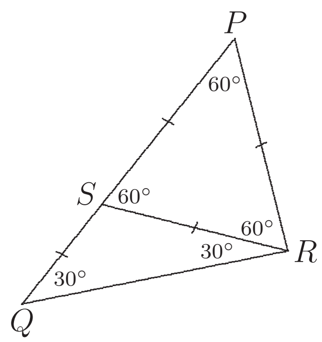
$$\frac{2}{3+2} \times 250 = \frac{2}{5} \times 250 = 100,$$

hence (B).

5. Since all numerators are 4, we need to choose the fraction with the smallest denominator and this is clearly $(0.4)^2 = 0.16$,

hence (C).

6. Since $\triangle PSR$ is equilateral, each angle is $60°$.



Since the area of $\triangle PRS$ is half that of $\triangle PQR$ it has the same area as $\triangle SRQ$. As these triangles have the same area and the same altitude from R to PQ, the bases PS and SQ are equal and hence $RS = SQ$.

Now $\angle QSR = 120°$ and $\angle SRQ = \angle SQR = 30°$. So $\angle PRQ = 60° + 30° = 90°$,

hence (C).

7. Let the radius of the quadrant be r. Then from Pythagoras' Theorem in $\triangle YOX, YX = \sqrt{r^2 + r^2} = r\sqrt{2}$.
$\mathsf{T} + \mathsf{S}$ is a quadrant of a circle radius r, so

$$\mathsf{T} + \mathsf{S} = \frac{1}{4}\pi r^2$$

$\mathsf{S} + \mathsf{C}$ is a semi-circle with a diameter of $r\sqrt{2}$, so

$$\mathsf{S} + \mathsf{C} = \frac{1}{8}\pi(r\sqrt{2})^2 = \frac{\pi r^2}{4}$$

Now

$$(\mathsf{T} + \mathsf{S}) - (\mathsf{C} + \mathsf{S}) = \mathsf{T} - \mathsf{C} = \frac{\pi r^2}{4} - \frac{\pi r^2}{4} = 0,$$

i.e. $\mathsf{T} = \mathsf{C}$, and thus the ratio

$$\text{area } \mathsf{T} : \text{ area } \mathsf{C} = 1,$$

hence (B).

Note: The same result is obtained by calculating the area of each of T and C to be $\frac{1}{2}r^2$.

8. Originally, 98% of the watermelon is water, so the solids are 2% or $\frac{1}{50}$ of its weight, i.e. $\frac{1}{50} \times 20 = \frac{2}{5}$ kg.

When the water becomes 95% of its weight, these solids become 5% or $\frac{1}{20}$ of its weight.

Thus its weight then is $20 \times \frac{2}{5} = 8\,\text{kg}$,

hence (E).

9. Alternative 1

The third entry in the first row must be 3, then the third column completes to 3, 2, 4, 5, 1 and the last row to 3, 5, 1, 2, 4. The first column can now be completed to 1, 4, 5, 2, 3. In the last column, 2 can only go in the third position, then that column completes to 2, 3, 1, 4, 5, so $x = 1$,

hence (A).

Alternative 2

The bottom left hand corner must be 3. Suppose the number in the indicated position is not 1, then it must be 3 as it has 2 above it, 5 underneath and 4 to the right. This then forces the number to the left of x to be 5, which then results in two 4s in the second row, which is impossible, thus x must be 1,

hence (A).

Note: This solution does not use the number 5 given in the middle column. If it were not given, there would be two ways to complete the square, though the number in the required position must still be 1. The reader is invited to verify this.

10. Since there are 8 teams, there are 7 rounds of four matches and thus a total of $7 \times 8 = 56$ points available.

Consider a team with 10 points. It is possible to have 5 teams on 10 points and 3 teams on 2 points when each of the top 5 draws with each other, each of the bottom 3 draws with each other and each of the top 5 wins against each of the bottom 3. So 10 points does not guarantee a place in the top 4.

Consider a team with 11 points. If this team was fifth then the number of points gained by the top 5 teams is ≥ 55. This is impossible as the number of points shared by the bottom 3 teams is then 1, as these 3 teams must have at least $3 \times 2 = 6$ points between them for the games played between themselves. Hence 11 points is sufficient to ensure a place in the top 4.

Thus 11 points are required,

hence (D).

Generalisation

Suppose we want to be guaranteed to be in the top k of n teams, for $1 \le k \le n-1$. With n teams there are $\binom{n}{n-1}$ games and hence $n(n-1) = n^2 - n$ points available.

(a) Consider a team with $2n - k - 2$ points.

Put the teams in two divisions A and B with $k + 1$ teams in Division A and $n - k - 1$ teams in Division B. Let all games within each division be ties (draws). All games across divisions are won by the teams in Division A. Then each team in Division A has $2n - k - 2$ points and none is guaranteed to be in the top k. So $2n - k - 2$ points is not enough.

(b) Consider a team with $2n - k - 1$ points.

There are $n^2 - n$ points altogether. Suppose at least $k+1$ teams have at least $2n - k - 1$ points each. Then together they have at least

$$(k + 1)(2n - k - 1) = 2nk + 2n - k^2 - 2k - 1$$

points.

This leaves $n^2 - 2nk - 3n + k^2 + 2k + 1$ points for the remaining $n - k - 1$ teams.

These teams must have between them at least

$$(n - k - 1)(n - k - 2) = n^2 - 2nk - 3n + k^2 + 3k + 2$$

points (for the $\binom{n-k-1}{n-k-2}$ games they have between themselves). This is a contradiction so at least one of the top $k+1$ teams must have less than $2n-k-1$ points and hence $2n - k - 1$ points guarantees a position in the top k teams.

Q10 is a particular case where $n = 8$ and $k = 4$. The required points are then $16 - 4 - 1 = 11$,

hence (D).